24/7/2021

Newton - Raphson's Method

Let 'x_0' be an approximate value of the root of the equation $f(x)=0$ which is either algebraic or transcendental. And let 'h' be a real number sufficiently small. If $\xi = x_0 + h$ be the exact root of $f(x)=0$, then

$$f(\xi)=0 \quad \text{or} \quad f(x_0+h)=0$$

Now expanding $f(x_0+h)$ by Taylor's series, we get

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots = 0$$

Since 'h' is very small, so neglecting the term containing h^2 and higher powers of 'h', we get

$$f(x_0) + hf'(x_0) \approx 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

Thus, the first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, taking x_1 as initial approximation, a still better approximation x_2 is obtained as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Continuing this process 'n' times, we get better approximation to the root which is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots$$

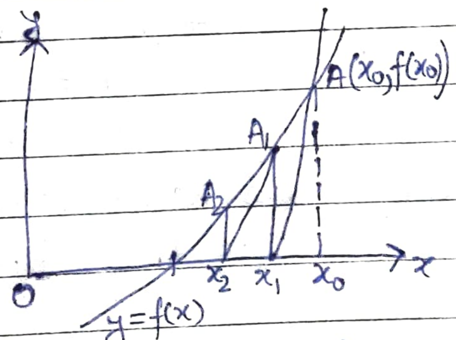
This is known as the Newton-Raphson formula or Newton's iteration formula.

Notes :-

1) If $f'(x)$ is large i.e., the graph of $y=f(x)$ is nearly vertical to the x-axis, while crossing it, then Newton's method is very useful.
2) If the initial approximation 'x_0' is so chosen sufficiently close to the root, then Newton-Raphson's method converges.
3) This method can also be used when the roots are complex.
4) Newton's method has a quadratic convergence.
5) The convergence of Newton's method for double root is linear.
6) Newton's method is conditionally convergent.
7) Newton's method converges if

$$|f(x)f''(x)| < |f'(x)|^2$$

Geometrical Interpretation of Newton-Raphson's Method.



Let 'x_0' be an approximation very close to the exact root ξ of the eqn $f(x)=0$. Let $A(x_0, f(x_0))$ be the point on curve $y=f(x)$.

Then the eqn of the tangent at $A(x_0, f(x_0))$ is given by $y-f(x_0) = f'(x_0)(x-x_0)$

This line cuts the x-axis when $y=0$, so that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The point 'x_1' is a first approximation to the exact root ξ. Let $A_1(x_1, (f(x_1)))$ be a point corresponding to x_1, then the tangent at A_1 cuts the x-axis at x_2 which is second approximation to the root ξ.

Newton's Formulae for Determining Special type of Roots

(i) Inverse of 'a' :-

The inverse of 'a' may be considered as a root of the equation

$$f(x) = \frac{1}{x} - a = 0 \quad \text{——— (1)}$$

$$\Rightarrow f'(x) = -\frac{1}{x^2}$$

Now, Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(\frac{1}{x_n} - a\right)}{\left(-1/x_n^2\right)}$$

$$\Rightarrow x_{n+1} = x_n(2 - a x_n) \quad \text{——— (2)}$$

Eqn (2) gives Newton's iterative formula for determining the approximate value of the inverse of 'a'.

(ii) Square root of 'a' :-

The square root of 'a' may be considered as a root of the equation

$$f(x) \equiv x^2 - a = 0 \quad \text{——— (3)}$$

$$\Rightarrow f'(x) = 2x$$

Now Newton-Raphson method becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^2 - a)}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \quad \text{——— (4)}$$

Eqn (4) gives the iterative formula for finding the square root of 'a'.

(iii) Inverse square root of 'a'.

The inverse square root of 'a' may be considered as a root of the equation.

$$f(x) \equiv \frac{1}{x^2} - a = 0$$

Thus the Newton-Raphson's method becomes

$$x_{n+1} = \frac{1}{2} x_n (3 - a x_n^2) \longrightarrow (5)$$

Eqn (5) gives Newton's iterative formula for finding the inverse square root of 'a'.

(iv) General formula for p^{th} root of 'a'.

The p^{th} root of 'a' may be considered as a root of the equation $f(x) \equiv x^p - a = 0$

$$\Rightarrow \quad f'(x) = p x^{p-1}$$

Now Newton-Raphson method becomes

$$x_{n+1} = x_n - \frac{(x_n^p - a)}{(p x_n^{p-1})}$$

$$\Rightarrow x_{n+1} = \frac{(p-1) x_n^p + a}{p x_n^{p-1}} \longrightarrow (6)$$

The eqn (6) gives the Newton's iterative formula for finding the p^{th} root of 'a'.

Similarly, the general formula for inverse of p^{th} root of 'a' is given by

$$x_{n+1} = x_n \left(\frac{p+1 - a x_n^p}{p} \right)$$